Schrodinger's Wave Equation.

In 1926, Schrodinger using de-Broglie's idea of Matter waves & developed a Mathematical formula which is known as wave-Mechanics dealing.

Let us Consider the vibration of a stretched string.

If 'ω' be the amplitude of any point whose Co-ordinate is 'x' at time 't'

The appropriate form of the wave equation may be written as follows

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 \omega}{\partial t^2} \quad \text{—— (I)}$$

Where 'u' is the velocity of Propagation of the wave on separating the variables, this differential equation may be written as

$$\omega = f(x) \cdot g(t) \quad \text{—— (ii)}$$

Where $f(x)$ is a function of the Co-ordinate 'x' only and $g(t)$ is a function of time 't' only.

For the motion of standing waves such as occuring in a stretched string, it is possible to express $g(t)$ as

$$g(t) = A \sin \omega t = A \sin 2\pi\nu t \quad \text{—— (iii)}$$

where ν is the vibrational frequency & A is constant and it stands for maximum amplitude.

From eq^n (ii) & (iii)

$$\omega = f(x) \cdot A \sin 2\pi\nu t$$

On differentiating the above equation with respect to 't'

we have,
$$\frac{\partial \omega}{\partial t} = f(x) \cdot A \cos 2\pi\nu t \cdot 2\pi\nu$$

$$\frac{\partial^2 \omega}{\partial t^2} = f(x)\,(-)\,A \sin 2\pi\nu t \cdot 2\pi\nu \cdot 2\pi\nu$$

or
$$\frac{\partial^2 \omega}{\partial t^2} = -4\pi^2\nu^2 f(x) \cdot A \sin 2\pi\nu t$$

$[\because A \sin 2\pi\nu t = g(t)]$

$$= -4\pi^2\nu^2 f(x)\, g(t) \quad \text{—— (iv)}$$

Now differentiating the eq^n (ii) with respect to x

$$\omega = f(x) \cdot g(t)$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial f(x)}{\partial x} \cdot g(t)$$

or
$$\frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^2 f(x)}{\partial x^2} \cdot g(t) \quad \text{—— (v)}$$

From equation (iv) and (v) i.e substituting the value of $\frac{\partial^2 \omega}{\partial t^2}$ and $\frac{\partial^2 \omega}{\partial x^2}$, in equation (i) i.e $\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \cdot \frac{\partial^2 \omega}{\partial t^2}$

$$\frac{\partial^2 f(x)}{\partial x^2} \cdot g(t) = \frac{1}{u^2} \left(-4\pi^2 \nu^2\right) f(x) \cdot g(t)$$

$$\text{or} \quad \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2 \nu^2}{u^2} f(x) \text{ --- (vi)}$$

But ν and u are related by the equation $u = \nu\lambda$ or $u^2 = \nu^2\lambda^2$, on Putting this value in the above eqn we have.

$$\therefore \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2 \cancel{\nu^2}}{\cancel{\nu^2}\lambda^2} f(x) \text{ ---}$$

$$\frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} f(x) \text{ --- (vii)}$$

It is the expression for the wave eqn in one direction and it can be extended in three directions, expressed by the Co-ordinate x, y and z. If $f(x)$ for one Co-ordinate is replaced by three Co-ordinate x, y & z i.e $\psi(x, y z)$, which is amplitude function for three Co-ordinates. then eqn (vii) take the form as follows.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2} \psi \text{ --- (viii)}$$

Using the Symbol ∇^2 for differential Operator

$$\text{i.e} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Here ∇^2 (Del squared) is known as Laplacian operator. then eqn (viii) may be replaced by

$$\nabla^2 \psi = -\frac{4\pi^2}{\lambda^2} \psi \text{ --- (ix)}$$

The above treatment is applicable to all particles including electrons, atoms and Photons.

By using, de-Broglie's relation $\lambda = \frac{h}{mu}$ in to eqn (ix)

we have

$$\nabla^2 \psi = -\frac{4\pi^2}{h^2} m^2 u^2 \psi \text{ --- (x)}$$

where m is mass of Particle, u is velocity and h is Plank Constant.

be Continued....

But we know that the total Energy of a Particle is the sum of its Potential energy and Kinetic energy

If, E = Total energy, U = Potential energy and Kinetic energy = $\frac{1}{2}m$

$$\therefore \quad E = KE + P.E$$

$$E = \frac{1}{2}mu^2 + U$$

$$\text{or } 2(E-U) = mu^2 \quad \text{———} (XI)$$

On Substituting the value of mu^2 in eqn (x)

we have

$$\nabla^2\psi = -\frac{4\pi^2}{h^2} m \cdot 2(E-U) \cdot \psi$$

$$\text{or } \nabla^2\psi = -\frac{8\pi^2}{h^2} m \, (E-U)\psi$$

$$\text{Or } \nabla^2\psi + \frac{8\pi^2 m}{h^2}(E-U)\psi = 0 \quad \text{—} (XII)$$

It is the required form of Schrodinger's Wave Equation.

—x—